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TOPOLOGIES ON DUAL SPACES AND SPACES OF LINEAR MAPPINGS

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ABSTRACT:

*Let X and Y be two **convex spaces** over the same (real or complex) field F. We consider a general method of defining **convex topologies** on the dual of a convex space, taking as neighborhoods of the origin the **polars** of certain sets in the convex space. Here we have proved that any finite sum of compact sets is compact. Also the sum of a compact and a closed set in a convex space is closed. Let V be the vector space of all continuous linear mappings of XintoY. Let A be any bounded subsets of X and B a base of **absolutely convex neighborhoods** in Y. Define $W_{A,B}$= {t: t (A) B} for each A A and B B. Then $W_{A,B}$is absolutely convex and **absorbent**. The topology is then called the **topology of A – convergence**. If X is a **barreled space**, then any point wise bounded set of continuous linear mappings of X into Y is**equicontinuous**.*

INTRODUCTION AND NECESSARY PRELIMINARIES

A ***topological vector space*** (tvs) is a set with two compatible structures, one, it has the algebraic structure of the vector space, and the other, it has a topology so that the notions of convergence and continuity are meaningful.

A subset V of a tvs X is ***convex*** if λ x + μ y $\in$ V for all x , y$\in$ V whenever $\lambda + \mu = 1$.

V is ***balanced***if x V for all x V whenever 1 . V is called ***absolutely convex*** if it is both convex and balanced. The set of all finite linear combinations $\sum_i \lambda_i x_i$ with

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$\lambda_i \geq 0, \sum \lambda^i = 1$ and each $x_i \in V$ is called ***convex envelope*** of A. The ***absolutely convex envelope*** of V is the set of all finite linear combinations $\sum_i \lambda_i x_i$ with $\sum_i |\lambda_i| \leq 1$ and each $x_i \in V$ and is the smallest convex set containing V. The set V is ***absorbent*** if x $\in$ V there is some $\lambda > 0$ such that x $\in \mu$V for all μ with $|\mu| \geq \lambda$.

A topology ξ on X is said to be ***compatible*** with the algebraic structure of X if the algebraic operations '+' and'.' are continuous in X.

A topological vector space is ***locally convex*** if it has a base of convex neighborhood of the origin.

In a convex space, a subset is called a ***barrel*** if it is absolutely convex, absorbent and closed. A convex space is called barreled if every barrel is a neighborhood.
A non-negative real valued function p on a tvs X is called a ***semi norm***
if (i) $p(x) \geq 0$;
(ii) $p(\lambda x) = |\lambda| p(x)$;
(iii) $p(x + y) \leq p(x) + p(y)$

2. General Method of Defining Convex Topologies and Compactness

Let (X, X′) be a dual pair and A any set of weakly bounded subsets of X. Then the sets A^0 (A $\in$ A) where is A^0 called a ***polar*** of A given by

$$A^0 = \{x' : \sup \{|\langle x, x' \rangle| : x \in A\} \leq 1\}$$

is absolutely convex and absorbent and so there is a coarsest topology τ' on X' in which they are neighborhoods. A base in neighborhoods in τ' is formed by the sets

$$\varepsilon \bigcap_{1 \le i \le n} A_i^0 = \left(\varepsilon^{-1} \bigcup_{1 \le i \le n} A_i\right)^0 \; \{\varepsilon > 0, A_i \in \mathrm{A}\}$$

This topology τ' on X' is called the topology of uniform convergence on the sets of $\mathcal{A}$ or the topology of $\mathcal{A}$- convergence or the ***polar topology***.

Now suppose X is a vector space over the field of real or complex numbers. Let U be a non-empty set of subsets of X such that

(i) if U $\in$ U , V$\in$ U , there is a W $\in$ U with W$\subseteq$U $\cap$V;
(ii) if U $\in$ U and $\alpha \neq 0, \alpha U \in$ U;
(iii) each U $\in$ U is absolutely convex and absorbent.

Then there exists a topology τ making X a convex space with U as a base of neighborhoods. This topology τ on X is called the ***convex topology***.

Theorem 2.1 Any finite sum of compact sets in a convex space is compact.

Proof: Let A and B be two compact sets and C an open covering of A + B. Then for each x $\in$ A and each y $\in$ B, there is an open absolutely convex neighborhood U(x,y) of the origin for which x + y + U(x,y) is contained in some set of C . Now keeping x fixed, the sets y + ½ U(x,y) form an open covering of B. As B is compact, let $\left\{y_j + \frac{1}{2}U(x, y_j) : 1 \le j \le n(x)\right\}$ be finite sub cover of y + ½ U(x,y). Let

$V(x) = \bigcap_{1 \le j \le n(x)} \frac{1}{2} U(x, y_j)$. Then the sets x +V(x) form an open covering of A. Again as

$\{x_k + V(x_k) : 1 \le k \le m\}$

A is compact, let

$\subseteq \bigcup_{\le \le} (\quad +$

$\subseteq \bigcup_{\le \le} \Big(\Big)$

$_i \quad V\, x_i \; : 1 \quad i \qquad m$ be a finite sub covering of A. Then

$$A + B \quad \square_{1\, i\, m} \; x_i \quad V\, x_i$$

$$\square_{\leq j\, n} \quad x_i + y_j + \frac{1}{2} U \quad , y_j \; + \frac{1}{2} U (x_i , y_j)$$

$$1\, i\, m\, 1 \qquad xi$$

$$\square \quad \square \, (x_i \quad y_j \quad U\, x_i , y_j$$

$$1\, i\, m\, 1$$

$$\subseteq \square \qquad + \quad + \quad (\qquad)$$

Hence A + B is compact.

Theorem2.2 The sum of a compact set and a closed set in a convex space is closed.

Proof: Let A be compact and B closed. Let $a \notin A + B$. Then for each x $\in$ A, x + B is closed, for + is continuous and B is closed. Hence there is an absolutely convex neighborhood U(x) of the origin with (a + U (x)) $\cap$(x + B) = ϕ. Then $a \notin x + U(x) + B$. Now $\left\{ x + \frac{1}{2} U (x) \right\}_{x \in A}$ form an open cover of A. Since A is compact, let $\left\{x_i + \frac{1}{2}U(x_i)\right\}_{x_i \in A}$, $1 \le i \le n$, form a finite sub cover of A. Let $V = \bigcap_{1 \le i \le n} \frac{1}{2}U(x_i)$.

Then

$$A + V \subseteq \bigcup_{1 \le i \le n}\left(x_i + \frac{1}{2}U(x_i) + \frac{1}{2}U(x_i)\right)$$

$$\subseteq \bigcup_{x \in A}\left(x + U(x)\right)$$

Hence $a \notin A + V + B$. Thus

(A+V)∩(A+B)= ϕ

and so $a \notin \overline{(A+B)}$. Hence A + B is closed.

If X is separated convex space, then the set of compact subsets of X can be used to define a polar topology on X′. Now if A is bounded, then the absolutely convex envelope of A is also bounded. However, it is not true in general that the closed absolutely convex envelope of a compact set is compact.

Remark: One may check whether the closed absolutely convex envelope of a super compact set is super compact or not?

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